Approximate Symmetry Reduction Approach: Infinite Series Reductions to the KdV-Burgers Equation

Xiaoyu Jiao[a], Ruoxia Yao[a,b,c], Shunli Zhang[d], and Sen Y. Lou[a,b]

[a] Department of Physics, Shanghai Jiao Tong University, Shanghai, 200240, China
[b] Department of Physics, Ningbo University, Ningbo, 315211, China
[c] School of Computer Science, Shaanxi Normal University, Xi'an, 710062, China
[d] Department of Mathematics, Northwest University, Xi'an, 710069, China

Reprint requests to Dr. X. J.; E-mail: jiaoxyxy@yahoo.com.cn

Z. Naturforsch. **64a,** 676 – 684 (2009); received November 8, 2008 / revised January 15, 2009

For weak dispersion and weak dissipation cases, the (1+1)-dimensional KdV-Burgers equation is investigated in terms of approximate symmetry reduction approach. The formal coherence of similarity reduction solutions and similarity reduction equations of different orders enables series reduction solutions. For the weak dissipation case, zero-order similarity solutions satisfy the Painlevé II, Painlevé I, and Jacobi elliptic function equations. For the weak dispersion case, zero-order similarity solutions are in the form of Kummer, Airy, and hyperbolic tangent functions. Higher-order similarity solutions can be obtained by solving linear variable coefficients ordinary differential equations.

Key words: KdV-Burgers Equation; Approximate Symmetry Reduction; Series Reduction Solutions
PACS number: 02.30.Jr

1. Introduction

Nonlinear problems arise in many fields of science and engineering. Lie group theory [1 – 3] greatly simplifies many nonlinear partial differential equations. Exact analytical solutions are nevertheless difficult to study in general. Perturbation theory [4 – 6] was thus developed and it plays an essential role in nonlinear science, especially in finding approximate analytical solutions to perturbed partial differential equations.

The integration of Lie group theory and perturbation theory yields two distinct approximate symmetry reduction methods. The first method due to Baikov et al. [7, 8] generalizes symmetry group generators to perturbation forms. For the second method proposed by Fushchich and Shtelen [9], dependent variables are expanded in perturbation series and the approximate symmetry of the original equation is decomposed into an exact symmetry of the system resulted from the perturbation. The second method is superior to the first one according to the comparison in [10, 11].

The well-known Korteweg-de Vries-Burgers (KdV-Burgers) equation

$$u_t + 6uu_x + \mu u_{xxx} + \nu u_{xx} = 0, \tag{1}$$

with μ and ν constant coefficients, is widely used in many physical fields especially in fluid dynamics. The effects of nonlinearity ($6uu_x$), dispersion (μu_{xxx}), and dissipation (νu_{xx}) are incorporated in this equation which simulates the propagation of waves on an elastic tube filled with a viscous fluid [12], the flow of liquids containing gas bubbles [13], and turbulence [14,15], etc.

Johnson [12] inspected the travelling wave solutions of the weak dissipation ($\nu \ll 1$) KdV-Burgers equation (1) in the phase plane by a perturbation method and developed a formal asymptotic expansion for the solution. Tanh function method was applied to (1) in the limit of weak dispersion ($\mu \ll 1$) in a perturbative way [16]. In [17, 18], perturbation analysis was also applied to the perturbed KdV equations

$$\eta_t + 6\eta\eta_x + \eta_{xxx} + \alpha c_1 \eta_{xx} + \alpha c_2 \eta_{xxxx} + \alpha c_3 (\eta\eta_x)_x = 0, \quad \alpha \ll 1, \tag{2}$$

and

$$u_t + uu_x + u_{xxx} = \varepsilon\alpha u + \varepsilon\beta u_{xx}, \quad \varepsilon \ll 1, \tag{3}$$

respectively.

Equation (1) can also be manipulated by means of the approximate symmetry reduction approach. Section 2 and Section 3 are devoted to apply this method to (1) under the case of weak dissipation ($\nu \ll 1$) and weak dispersion ($\mu \ll 1$), respectively. Section 4 includes conclusion and discussion of the results.

2. Approximate Symmetry Reduction Approach to Weak Dissipation KdV-Burgers Equation

According to the perturbation theory, solutions of perturbed partial differential equations can be expressed as series solutions with respect to all powers of a small parameter. Specifically, we suppose that the weak dissipation ($\nu \ll 1$) KdV-Burgers equation (1) has the solution

$$u = \sum_{k=0}^{\infty} \nu^k u_k, \tag{4}$$

where u_k are functions of x and t, and solve the following system

$$u_{k,t} + 6\sum_{i=0}^{k} u_{k-i}u_{i,x} + \mu u_{k,xxx} + u_{k-1,xx} = 0, \quad (k = 0,1,\cdots) \tag{5}$$

with $u_{-1} = 0$, which is obtained by inserting (4) into (1) and vanishing the coefficients of different powers of ν.

The next crucial step is to study the symmetry reduction of the above system via the Lie symmetry approach [19]. For this purpose, we construct the Lie point symmetry transformations

$$\sigma_k = Xu_{k,x} + Tu_{k,t} - U_k, \quad (k = 0,1,\cdots), \tag{6}$$

where X, T, and U_k are functions with respect to x, t, and u_i, $(i = 0,1,\cdots)$. The linearized equations for (5) are

$$\sigma_{k,t} + 6\sum_{i=0}^{k} (\sigma_{k-i}u_{i,x} + u_{k-i}\sigma_{i,x}) + \mu\sigma_{k,xxx} + \sigma_{k-1,xx} = 0, \quad (k = 0,1,\cdots) \tag{7}$$

with $\sigma_{-1} = 0$. (7) means that (5) is invariant under the transformations $u_k \to u_k + \varepsilon\sigma_k$ $(k = 0,1,\cdots)$ with an infinitesimal parameter ε.

There exist an infinite number of equations (5) and (7) and an infinite number of arguments in X, T, and U_k $(k = 0,1,\cdots)$. To simplify the problem, we start the discussion for a finite number of equations.

Confining the range of k to $\{k|k = 0,1,2\}$ in (5), (6), and (7), we see that X, T, U_0, U_1, and U_2 are functions with respect to x, t, u_0, u_1, and u_2. In this case, the determining equations can be obtained by substituting (6) into (7), eliminating $u_{0,t}$, $u_{1,t}$, and $u_{2,t}$ in terms of (5) and vanishing all coefficients of different partial derivatives of u_0, u_1, and u_2. Some of the determining equations read

$$T_x = T_{u_0} = T_{u_1} = T_{u_2} = 0,$$

from which we have $T = T(t)$. Considering this condition, we choose the simplest equations for X:

$$X_{u_0} = X_{u_1} = X_{u_2} = 0,$$

from which we have $X = X(x,t)$. Considering this condition, we choose the simplest equations for U_0, U_1, and U_2:

$$\begin{aligned} U_{0,xu_2} &= U_{0,u_0u_0} = U_{0,u_0u_1} = U_{0,u_0u_2} \\ &= U_{0,u_1u_1} = U_{0,u_1u_2} = U_{0,u_2u_2} = 0, \end{aligned}$$

$$\begin{aligned} U_{1,u_0u_0} &= U_{1,u_0u_1} = U_{1,u_0u_2} = U_{1,u_1u_1} \\ &= U_{1,u_1u_2} = U_{1,u_2u_2} = 0, \end{aligned}$$

$$\begin{aligned} U_{2,u_0u_0} &= U_{2,u_0u_1} = U_{2,u_0u_2} = U_{2,u_1u_1} \\ &= U_{2,u_1u_2} = U_{2,u_2u_2} = 0 \end{aligned}$$

of which the solutions are

$$U_0 = F_1(x,t)u_0 + F_2(x,t)u_1 + F_3(t)u_2 + F_4(x,t),$$

$$U_1 = F_5(x,t)u_0 + F_6(x,t)u_1 + F_7(x,t)u_2 + F_8(x,t),$$

$$U_2 = F_9(x,t)u_0 + F_{10}(x,t)u_1 + F_{11}(x,t)u_2 + F_{12}(x,t).$$

Under these relations, the determining equations are simplified to

$$\begin{aligned} X_{xx} &= F_2 = F_3 = F_5 = F_7 = F_8 = F_9 = F_{10} = F_{12} \\ &= F_{1,x} = F_{4,xx} = F_{4,t} = F_{6,x} = F_{11,x} = 0, \end{aligned}$$

$$T_t = 3X_x, \quad X_t = 6F_4, \quad F_{1,t} = -6F_{4,x},$$
$$F_{6,t} = -6F_{4,x}, \quad F_{11,t} = -6F_{4,x},$$

$$T_t = X_x - F_1, \quad T_t = 2X_x - F_1 + F_6,$$
$$T_t = X_x + F_{11} - 2F_6, \quad T_t = 2X_x - F_6 + F_{11},$$

which provide us with

$$X = 6at + cx + x_0, \quad T = 3ct + t_0,$$
$$U_0 = -2cu_0 + a, \quad U_1 = -cu_1, \quad U_2 = 0,$$

where a, c, x_0, and t_0 are arbitrary constants.

Similarly, limiting the range of k to $\{k|k=0,1,2,3\}$ in (5), (6), and (7), where X, T, U_0, U_1, U_2, and U_3 are functions with respect to x, t, u_0, u_1, u_2, and u_3, we repeat the calculation as before and obtain:

$$X=6at+cx+x_0,\quad T=3ct+t_0,\quad U_0=-2cu_0+a,$$
$$U_1=-cu_1,\quad U_2=0,\quad U_3=cu_3,$$

where a, c, x_0, and t_0 are arbitrary constants.

With more similar calculation considered, we see that X, T, and U_k $(k=0,1,\cdots)$ are formally coherent, i. e.,

$$\begin{aligned} &X=6at+cx+x_0,\quad T=3ct+t_0,\\ &U_k=(k-2)cu_k+a\delta_{k,0},\quad (k=0,1,\cdots), \end{aligned} \tag{8}$$

where a, c, x_0, and t_0 are arbitrary constants. The notation $\delta_{k,0}$ satisfying $\delta_{0,0}=1$ and $\delta_{k,0}=0$ $(k\neq 0)$ is adopted in the following text. Subsequently, solving the characteristic equations

$$\begin{aligned} &\frac{\mathrm{d}x}{X}=\frac{\mathrm{d}t}{T},\quad \frac{\mathrm{d}u_0}{U_0}=\frac{\mathrm{d}t}{T},\quad \frac{\mathrm{d}u_1}{U_1}=\frac{\mathrm{d}t}{T},\quad \cdots,\\ &\frac{\mathrm{d}u_k}{U_k}=\frac{\mathrm{d}t}{T},\quad \cdots \end{aligned} \tag{9}$$

leads to the similarity solutions of (5) which can be distinguished in the following two subcases.

2.1. Symmetry Reduction of the Painlevé II Solutions

If $c\neq 0$, for brevity of the results, we rewrite the constants a, x_0, and t_0 as ca, cx_0, and ct_0, respectively. Solving $\frac{\mathrm{d}x}{X}=\frac{\mathrm{d}t}{T}$ in (9) leads to the invariant

$$I(x,t)=\xi=(x-3at+x_0-3at_0)(3t+t_0)^{-\frac{1}{3}}. \tag{10}$$

In the same way, we get other invariants

$$I_0(x,t,u_0)=P_0=\frac{1}{2}(3t+t_0)^{\frac{2}{3}}(2u_0-a) \tag{11}$$

and

$$I_k(x,t,u_k)=P_k=u_k(3t+t_0)^{-\frac{1}{3}(k-2)},\quad (k=1,2,\cdots) \tag{12}$$

from $\frac{\mathrm{d}u_0}{U_0}=\frac{\mathrm{d}t}{T}$, and $\frac{\mathrm{d}u_k}{U_k}=\frac{\mathrm{d}t}{T}$ $(k=1,2,\cdots)$, respectively. Viewing P_k $(k=0,1,\cdots)$ as functions of ξ, we get the similarity solutions to (5):

$$u_k=(3t+t_0)^{\frac{1}{3}(k-2)}P_k(\xi)+\frac{1}{2}a\delta_{k,0},\quad (k=0,1,\cdots) \tag{13}$$

with the similarity variable

$$\xi=(x-3at+x_0-3at_0)(3t+t_0)^{-\frac{1}{3}}. \tag{14}$$

Accordingly, the series reduction solution of (1) is derived

$$u=\frac{a}{2}+\sum_{k=0}^{\infty}\nu^k(3t+t_0)^{\frac{1}{3}(k-2)}P_k(\xi), \tag{15}$$

and the related similarity reduction equations are

$$\begin{aligned} &\mu P_{k,\xi\xi\xi}+6\sum_{i=0}^{k}P_{k-i}P_{i,\xi}-\xi P_{k,\xi}+(k-2)P_k\\ &\qquad +P_{k-1,\xi\xi}=0,\quad (k=0,1,\cdots) \end{aligned} \tag{16}$$

with $P_{-1}=0$. If $k=0$, (16) is equivalent to the Painlevé II type equation. The nth $(n>0)$ similarity reduction equation is actually a third order *linear* variable coefficients ordinary differential equation of P_n if the previous $P_0, P_1, \cdots, P_{n-1}$ are known, since (16) is just

$$\begin{aligned} &\mu P_{k,\xi\xi\xi}+6(P_0P_{k,\xi}+P_kP_{0,\xi})-\xi P_{k,\xi}\\ &+(k-2)P_k=f_k(\xi),\quad (k=1,2,\cdots), \end{aligned} \tag{17}$$

where f_k is only a function of $\{P_0,P_1,\cdots,P_{k-1}\}$:

$$f_k(\xi)=-6\sum_{i=1}^{k-1}P_{k-i}P_{i,\xi}-P_{k-1,\xi\xi}. \tag{18}$$

To see the behaviour of the series solution (15), we simply take the first six polynomial solutions of (16):

$$\begin{aligned} &P_0=P_1=0,\ P_2=c_{2,0},\ P_3=c_{3,1}\xi,\ P_4=c_{4,2}\xi^2,\\ &P_5=c_{5,0}-\frac{1}{6\mu}(2c_{4,2}+6c_{2,0}c_{3,1}+3c_{5,0})\xi^3, \end{aligned} \tag{19}$$

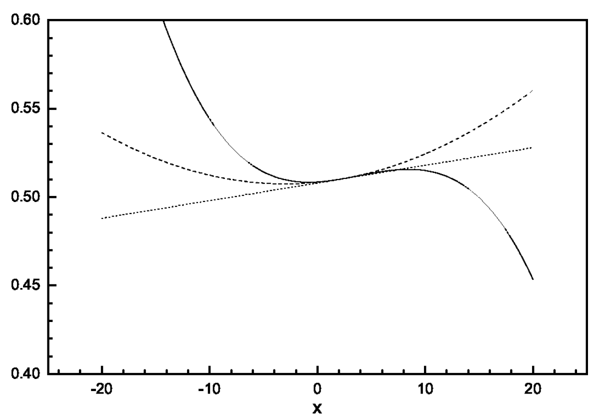


Fig. 1. Plots of truncated series solutions $\sum_{i=0}^{k}u_i$ of (13) and (19) for $k=3$ (dotted line), $k=4$ (dashed line), and $k=5$ (solid line) with $\nu=0.1$, $a=x_0=t_0=\mu=c_{2,0}=c_{3,1}=c_{4,2}=c_{5,0}=1$, and $t=0$.

where $c_{2,0}$, $c_{3,1}$, $c_{4,2}$, and $c_{5,0}$ are arbitrary constants. In Figure 1, the truncated series solutions $\sum_{i=0}^{3} u_i$ (dotted line), $\sum_{i=0}^{4} u_i$ (dashed line), and $\sum_{i=0}^{5} u_i$ (solid line) are specified by taking $\nu = 0.1$, $a = x_0 = t_0 = \mu = c_{2,0} = c_{3,1} = c_{4,2} = c_{5,0} = 1$, $t = 0$ of (13) and (19). If t is fixed, infinite point is an essential singular point of the series solution (15) which is divergent. As t increases, the convergence is even worse, since the factor $(3t+t_0)^{\frac{1}{3}(k-2)}$ in the series solution grows dramatically if k is large enough.

2.2. Symmetry Reduction of the Painlevé I Solutions

If $c = 0$ and $t_0 \neq 0$, we rewrite a and x_0 as at_0 and x_0t_0, respectively. The similarity solutions are

$$u_k = (at + \frac{x_0}{6})\delta_{k,0} + P_k(\xi), \quad (k = 0, 1, \cdots), \tag{20}$$

with $\xi = -x + 3at^2 + x_0t$. Hence, the series reduction solution of (1) is

$$u = at + \frac{x_0}{6} + \sum_{k=0}^{\infty} \nu^k P_k(\xi), \tag{21}$$

in which $P_k(\xi)$ satisfies

$$\mu P_{k,\xi\xi} + 3\sum_{i=0}^{k} P_{k-i}P_i - P_{k-1,\xi} - a\delta_{k,0}\xi + A_k = 0, \quad (k = 0, 1, \cdots), \tag{22}$$

where $P_{-1} = 0$ and A_k are arbitrary integral constants.

If $k = 0$, (22) is equivalent to the Painlevé I type equation provided that $a \neq 0$.

If $k = 0$, $a = 0$, and $A_0 = \frac{4}{3}\mu^2 p_1^4(m^2 - 1 - m^4)$, (22) can be solved by the Jacobi elliptic function

$$P_0 = \frac{2}{3}\mu p_1^2(1 - 2m^2) + 2\mu m^2 p_1^2 \mathrm{cn}^2(p_1\xi + p_2, m), \tag{23}$$

where p_1, p_2, and m are arbitrary constants.

If $k > 0$, an equivalent form of (22) is

$$\mu P_{k,\xi\xi} + 6P_kP_0 = g_k(\xi), \quad (k = 1, 2, \cdots), \tag{24}$$

where $g_k(\xi)$ is a function of $\{P_0, P_1, \cdots, P_{k-1}\}$ as follows:

$$g_k(\xi) = -3\sum_{i=1}^{k-1} P_{k-i}P_i + P_{k-1,\xi} - A_k. \tag{25}$$

From (24), we see that (22) is a second-order linear variable coefficients ordinary differential equation of P_k and can be integrated out step by step if $a = 0$. The results read

$$P_k = P_{0,\xi}\left[C_k + \mu^{-1}\int P_{0,\xi}^{-2}\left(B_k + \int P_{0,\xi}g_k \mathrm{d}\xi\right)\mathrm{d}\xi\right], \tag{26}$$

with arbitrary integral constants B_k and C_k.

The compact form of the general solution above is nevertheless not capable of producing a formal compact solution of (22), since the computation of integration often fails. We just find the first four polynomial solutions of (22):

$$\begin{aligned}
&P_0 = 0,\\
&P_1 = c_{1,0} + c_{1,1}\xi + c_{1,2}\xi^2,\\
&P_2 = c_{2,0} + c_{2,1}\xi + c_{2,2}\xi^2 + \frac{1}{3\mu}(c_{1,2} - 3c_{1,0}c_{1,1})\xi^3\\
&-\frac{1}{4\mu}(c_{1,1}^2 + 2c_{1,0}c_{1,2})\xi^4 - \frac{3}{10\mu}c_{1,1}c_{1,2}\xi^5 - \frac{1}{10\mu}c_{1,2}^2\xi^6,\\
&P_3 = c_{3,0} + c_{3,1}\xi + c_{3,2}\xi^2 + \frac{1}{3\mu}\Big(c_{2,2} - 3c_{1,1}c_{2,0}\\
&-3c_{1,0}c_{2,1}\Big)\xi^3 - \frac{1}{12\mu^2}\Big(6c_{1,0}c_{2,2}\mu + 3c_{1,0}c_{1,1}\\
&+6c_{1,1}c_{2,1}\mu + 6c_{1,2}c_{2,0}\mu - c_{1,2}\Big)\xi^4 - \frac{1}{20\mu^2}\Big(6c_{1,1}c_{2,2}\mu\\
&+c_{1,1}^2 + 4c_{1,0}c_{1,2} + 6c_{1,2}c_{2,1}\mu - 6c_{1,0}^2c_{1,1}\Big)\xi^5\\
&-\frac{1}{60\mu^2}\Big(7c_{1,1}c_{1,2} - 15c_{1,0}c_{1,1}^2 - 6c_{1,0}^2c_{1,2}\\
&+12c_{1,2}c_{2,2}\mu)\xi^6 + \frac{1}{420\mu^2}\Big(15c_{1,1}^3 + 108c_{1,0}c_{1,1}c_{1,2}\\
&-26c_{1,2}^2\Big)\xi^7 + \frac{3}{560\mu^2}c_{1,2}\Big(12c_{1,0}c_{1,2} + 11c_{1,1}^2\Big)\xi^8\\
&+\frac{1}{30\mu^2}c_{1,1}c_{1,2}^2\xi^9 + \frac{1}{150\mu^2}c_{1,2}^3\xi^{10},
\end{aligned} \tag{27}$$

if $a = A_0 = 0, A_1 = -2c_{1,2}\mu, A_2 = c_{1,1} - 3c_{1,0}^2 - 2c_{2,2}\mu$, $A_3 = c_{2,1} - 2c_{3,2}\mu - 6c_{1,0}c_{2,0}$, and $c_{1,0}$, $c_{1,1}$, $c_{1,2}$, $c_{2,0}$, $c_{2,2}$, $c_{3,0}$, $c_{3,1}$, and $c_{3,2}$ are arbitrary constants.

From (20) and (27), taking $\nu = 0.1$, $x_0 = \mu = c_{1,0} = c_{1,1} = c_{1,2} = c_{2,0} = c_{2,1} = c_{2,2} = c_{3,0} = c_{3,1} = c_{3,2} = 1$, we construct the truncated series solutions $u_0 + u_1$, $u_0 + u_1 + u_2$, and $u_0 + u_1 + u_2 + u_3$, which are displayed for $-5 < x < 6$ and $t = 0$ in Figure 2, where the dotted line, the dashed line, and the solid line represent $u_0 + u_1$, $u_0 + u_1 + u_2$, and $u_0 + u_1 + u_2 + u_3$, respectively. It is easily seen that the superposition fraction

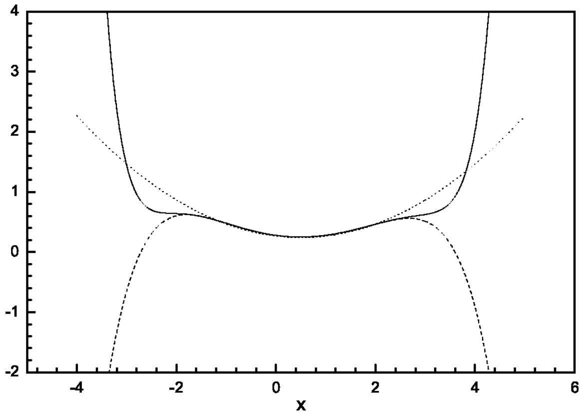


Fig. 2. Plots of truncated series solutions u_0+u_1, $u_0+u_1+u_2$, and $u_0+u_1+u_2+u_3$ of (20) and (27) with $\nu=0.1$, $x_0=\mu=c_{1,0}=c_{1,1}=c_{1,2}=c_{2,0}=c_{2,1}=c_{2,2}=c_{3,0}=c_{3,1}=c_{3,2}=1$, and $a=t=0$.

for the curves of truncated series solutions $u_0+u_1+u_2$ and $u_0+u_1+u_2+u_3$ is larger than that of u_0+u_1 and $u_0+u_1+u_2$, implying that more terms in the truncated series solutions means better approximation of the exact solutions. Infinite point is an essential singular point of the series solution (21) if t is specified.

3. Approximate Symmetry Reduction Approach to Weak Dispersion KdV-Burgers Equation

We search for series reduction solutions to weak dispersion $(\mu \ll 1)$ KdV-Burgers equation (1). The process is similar to Section 2. A system of partial differential equations

$$u_{k,t}+6\sum_{i=0}^{k}u_{k-i}u_{i,x}+\nu u_{k,xx}+u_{k-1,xxx}=0, \quad (k=0,1,\cdots), \tag{28}$$

with $u_{-1}=0$, is obtained by plugging the perturbation series solution

$$u=\sum_{k=0}^{\infty}\mu^k u_k \tag{29}$$

into (1) and vanishing the coefficients of different powers of μ.

The linearized equations related to (28) are

$$\sigma_{k,t}+6\sum_{i=0}^{k}(\sigma_{k-i}u_{i,x}+u_{k-i}\sigma_{i,x})+\nu\sigma_{k,xx}+\sigma_{k-1,xxx}=0, \quad (k=0,1,\cdots), \tag{30}$$

with $\sigma_{-1}=0$. The Lie point symmetry transformations (6) satisfy the above linearized equations under the approximate equations (28). Restricting the range of k to $\{k|k=0,1,2\}$ in (6), (28), and (30), we see that X, T, U_0, U_1, and U_2 are functions with respect to x, t, u_0, u_1, and u_2. The determining equations are derived by substituting (6) into (30), eliminating $u_{0,t}$, $u_{1,t}$, and $u_{2,t}$ in terms of (28) and vanishing coefficients of different partial derivatives of u_0, u_1, and u_2. Some of the determining equations are

$$T_x=T_{u_0}=T_{u_1}=T_{u_2}=0,$$

from which we get $T=T(t)$. Considering this condition, the simplest equations for X in the determining equations are

$$X_{u_0}=X_{u_1}=X_{u_2}=0,$$

from which we get $X=X(x,t)$. Considering this condition, we select the simplest equations for U_0 and U_1:

$$U_{0,u_0u_0}=U_{0,u_0}=U_{0,u_2}=0,$$
$$U_{1,xu_0}=U_{1,u_0u_0}=U_{1,u_0u_1}=U_{1,u_1u_1}=U_{1,u_2}=0,$$

with the solution $U_0=F_1(x,t)u_0+F_2(x,t)$ and $U_1=F_3(t)u_0+F_4(x,t)u_1+F_5(x,t)$. In the reduced determining equations, the simplest equations for U_2 read as

$$U_{2,u_0u_0}=U_{2,u_0u_1}=U_{2,u_0u_2}=U_{2,u_1u_1}=U_{2,u_1u_2}=U_{2,u_2u_2}=0,$$

leading to $U_2=F_6(x,t)u_0+F_7(x,t)u_1+F_8(x,t)u_2+F_9(x,t)$.

Combined with these conditions, the determining equations are simplified to

$$X_{xx}=F_3=F_5=F_6=F_7=F_9=F_{1,x}=F_{2,xxx}=F_{4,x}=F_{8,x}=0,$$

$$T_t=2X_x,\quad X_t=6F_2,\quad F_{1,t}=-6F_{2,x},$$
$$F_{2,t}=-\nu F_{2,xx},\quad F_{4,t}=-6F_{2,x},\quad F_{8,t}=-6F_{2,x},$$
$$T_t=X_x-F_1,\quad T_t=X_x-2F_4+F_8,$$
$$T_t=3X_x-F_1+F_4,\quad T_t=3X_x-F_4+F_8.$$

It is easily seen that

$$X=6at+cx+x_0,\quad T=2ct+t_0,\quad U_0=-cu_0+a,$$
$$U_1=-2cu_1,\quad U_2=-3cu_2,$$

where a, c, x_0, and t_0 are arbitrary constants.

In the same manner, we obtain

$$X = 6at + cx + x_0, \quad T = 2ct + t_0, \quad U_0 = -cu_0 + a,$$
$$U_1 = -2cu_1, \quad U_2 = -3cu_2, \quad U_3 = -4cu_3,$$

where a, c, x_0, and t_0 are arbitrary constants.

Repeating similar calculation several times, we summarize the solutions of the determining equations

$$\begin{aligned} &X = 6at + cx + x_0, \quad T = 2ct + t_0, \\ &U_k = -(k+1)cu_k + a\delta_{k,0}, \quad (k = 0, 1, \cdots), \end{aligned} \tag{31}$$

where a, c, x_0, and t_0 are arbitrary constants. The similarity solutions of (28) from solving the characteristic equations (9) are discussed in the following two subcases.

3.1. Symmetry Reduction of the Kummer Function Solutions

If $c \neq 0$, for brevity of the results, we rewrite the constants a, x_0, and t_0 as ca, cx_0, and ct_0, respectively. Solving $\mathrm{d}x/X = \mathrm{d}t/T$ in (9) results in the invariant

$$I(x,t) = \xi = (x - 6at - 6at_0 + x_0)(2t + t_0)^{-\frac{1}{2}}. \tag{32}$$

Likewise, we get other invariants

$$I_0(x,t,u_0) = P_0 = (2t + t_0)^{\frac{1}{2}}(u_0 - a) \tag{33}$$

and

$$I_k(x,t,u_k) = P_k = u_k(2t + t_0)^{\frac{1}{2}(k+1)}, \ (k = 1, 2, \cdots), \tag{34}$$

from $\mathrm{d}u_0/U_0 = \mathrm{d}t/T$ and $\mathrm{d}u_k/U_k = \mathrm{d}t/T$ $(k = 1, 2, \cdots)$, respectively. Consequently, we get the similarity solutions of (28)

$$u_k = (2t + t_0)^{-\frac{1}{2}(k+1)} P_k(\xi) + a\delta_{k,0}, \ (k = 0, 1, \cdots) \tag{35}$$

with $\xi = (x - 6at - 6at_0 + x_0)(2t + t_0)^{-\frac{1}{2}}$, and the series reduction solution of (1) is

$$u = a + \sum_{k=0}^{\infty} \mu^k (2t + t_0)^{-\frac{1}{2}(k+1)} P_k(\xi), \tag{36}$$

where $P_k(\xi)$ conform to

$$\begin{aligned} &\nu P_{k,\xi\xi\xi} + 6\sum_{i=0}^{k} P_{k-i}P_{i,\xi} - \xi P_{k,\xi} - (k+1)P_k \\ &+ P_{k-1,\xi\xi\xi} = 0, \quad (k = 0, 1, \cdots), \end{aligned} \tag{37}$$

with $P_{-1} = 0$.

If $k = 0$, (37) has the Kummer function solution

$$\begin{aligned} P_0 = &\left\{(3C_1 - 1)\nu\left[3C_1C_2\mathrm{K}_1\left(\frac{3}{2}(1 - C_1), \frac{3}{2}, \frac{\xi^2}{2\nu}\right)\right.\right. \\ &\left.\left. - 2\mathrm{K}_2\left(\frac{3}{2}(1 - C_1), \frac{3}{2}, \frac{\xi^2}{2\nu}\right)\right]\right\} \Big/ \\ &\left\{6\xi\left[C_2\mathrm{K}_1\left(\frac{1}{2}(1 - 3C_1), \frac{3}{2}, \frac{\xi^2}{2\nu}\right)\right.\right. \\ &\left.\left. + \mathrm{K}_2\left(\frac{1}{2}(1 - 3C_1), \frac{3}{2}, \frac{\xi^2}{2\nu}\right)\right]\right\} + \frac{C_1\nu}{\xi}, \end{aligned} \tag{38}$$

where C_1 and C_2 are arbitrary constants, and the two types of Kummer functions $\mathrm{K}_1(p,q,z)$ and $\mathrm{K}_2(p,q,z)$ solve the differential equation

$$zy''(z) + (q - z)y'(z) - py(z) = 0.$$

If $k > 0$, we rearrange the terms in (37) as

$$\begin{aligned} &\nu P_{k,\xi\xi\xi} + 6(P_0P_{k,\xi} + P_kP_{0,\xi}) - \xi P_{k,\xi} - (k+1)P_k \\ &= -6\sum_{i=1}^{k-1} P_{k-i}P_{i,\xi} - P_{k-1,\xi\xi\xi}, \quad (k = 1, 2, \cdots), \end{aligned} \tag{39}$$

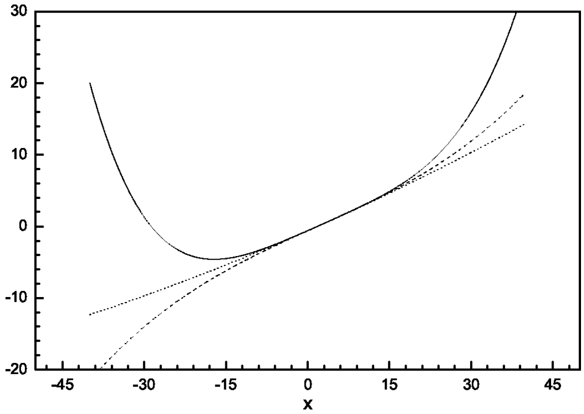
which is a second order linear variable coefficients ordinary differential equation of P_k if the previous P_0, P_1, $\cdots$, P_{k-1} are known.

The first six polynomial solutions of (37) are

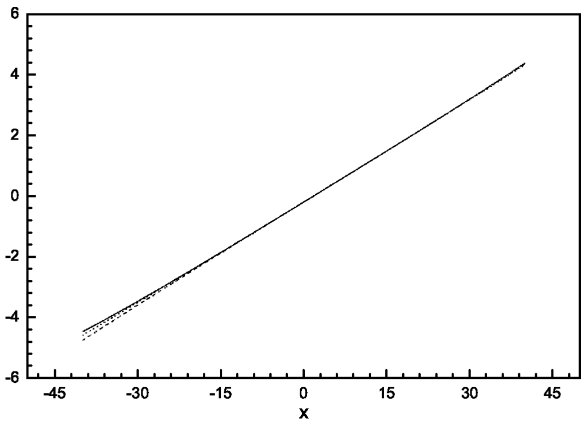
$$\begin{aligned} &P_0 = \frac{\xi}{3}, \quad P_1 = c_{1,0}, \quad P_2 = c_{2,1}\xi, \\ &P_3 = c_{3,2}\nu + 3c_{1,0}c_{2,1} + c_{3,2}\xi^2, \\ &P_4 = 3(c_{4,3}\nu + 2c_{1,0}c_{3,2} + c_{2,1}^2)\xi + c_{4,3}\xi^3, \\ &P_5 = 9c_{1,0}c_{2,1}^2 + 9c_{1,0}^2c_{3,2} + \frac{3}{2}c_{4,3} + 3c_{5,4}\nu^2 \\ &\quad + (9c_{1,0}c_{4,3} + 6c_{2,1}c_{3,2})\nu \\ &\quad + (9c_{1,0}c_{4,3} + 9c_{2,1}c_{3,2} + 6c_{5,4}\nu)\xi^2 + c_{5,4}\xi^4, \end{aligned} \tag{40}$$

where $c_{1,0}$, $c_{2,1}$, $c_{3,2}$, $c_{4,3}$, and $c_{5,4}$ are arbitrary constants.

From (35) and (40), taking $\mu = 0.1$, $a = \nu = x_0 = t_0 = c_{1,0} = c_{2,1} = c_{3,2} = c_{4,3} = c_{5,4} = 1$, we construct truncated series solutions $\sum_{i=0}^{3} u_i$, $\sum_{i=0}^{4} u_i$, and $\sum_{i=0}^{5} u_i$, which are displayed for $-50 < x < 50$ and $t = 0, 1$ in Figure 3, where the dotted line, the dashed line, and the solid line correspond to $\sum_{i=0}^{3} u_i$, $\sum_{i=0}^{4} u_i$, and $\sum_{i=0}^{5} u_i$, respectively. In Figure 3, the superposition fraction for the curves of truncated series solutions grows if t goes from 0 to 1, in other words, the convergence region of the series solution (36) extends as t increases, which is



$t=0$



$t=1$

Fig. 3. Plots of truncated series solutions $\sum_{i=0}^{k} u_i$ from (35) and (40) for $\{k|k=3,4,5\}$ with $\mu=0.1$, $a=\nu=x_0=t_0=c_{1,0}=c_{2,1}=c_{3,2}=c_{4,3}=c_{5,4}=1$.

due to the fact that the powers of the terms containing t in the truncated series solutions are nonpositive. The singularity of infinite point for the series solution (36) is an essential singularity if we fix t.

3.2. Symmetry Reduction of Airy Function and Hyperbolic Tangent Function Solutions

If $c=0$ and $t_0\neq 0$, we rewrite the constants a and x_0 as at_0 and x_0t_0, respectively. It is easily seen that the similarity solutions are

$$u_k=(at+\frac{x_0}{6})\delta_{k,0}+P_k(\xi),\quad (k=0,1,\cdots),\tag{41}$$

with $\xi=-x+3at^2+x_0t$, and the series reduction solution of (1) is

$$u=at+\frac{x_0}{6}+\sum_{k=0}^{\infty}\mu^kP_k(\xi)\tag{42}$$

with $P_k(\xi)$ satisfying

$$\nu P_{k,\xi}-3\sum_{i=0}^{k}P_{k-i}P_i-P_{k-1,\xi\xi}+a\delta_{k,0}\xi+A_k=0,\quad (k=0,1,\cdots),\tag{43}$$

where $P_{-1}=0$ and A_k are integral constants.

If $k=0$ and $a\equiv -b$, we get the Airy function solution of (43)

$$\begin{aligned}P_0=\Big\{&(3b\nu)^{\frac{1}{3}}\Big[C_1\mathrm{Ai}\Big(1,3^{\frac{1}{3}}(b\nu)^{-\frac{2}{3}}(A_0-b\xi)\Big)\\&+\mathrm{Bi}\Big(1,3^{\frac{1}{3}}(b\nu)^{-\frac{2}{3}}(A_0-b\xi)\Big)\Big]\Big\}\Big/\\&\Big\{3\Big[C_1\mathrm{Ai}\Big(3^{\frac{1}{3}}(b\nu)^{-\frac{2}{3}}(A_0-b\xi)\Big)\\&+\mathrm{Bi}\Big(3^{\frac{1}{3}}(b\nu)^{-\frac{2}{3}}(A_0-b\xi)\Big)\Big]\Big\},\end{aligned}\tag{44}$$

where C_1 is an arbitrary constant. The Airy wave functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are linearly independent solutions for $y(z)$ in the equation $y''(z)-zy(z)=0$. $\mathrm{Ai}(n,z)$ and $\mathrm{Bi}(n,z)$ are the nth derivatives of $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ evaluated at z, respectively.

The hyperbolic tangent function solution of travelling wave form

$$P_0=-\frac{\sqrt{3}}{3}p\tanh[\frac{\sqrt{3}}{\nu}p(\xi+d)]\tag{45}$$

with d an arbitrary constant, can be obtained from $a=0$ and $p=\sqrt{A_0}$.

If $k>0$, an equivalent form of (43) is

$$\nu P_{k,\xi}-6P_0P_k=g_k(\xi),\quad (k=1,2,\cdots),\tag{46}$$

where

$$g_k(\xi)=3\sum_{i=1}^{k-1}P_{k-i}P_i+P_{k-1,\xi\xi}-A_k.\tag{47}$$

From (46), it is easily seen that the kth similarity reduction equation in (43) is a first order linear variable coefficients ordinary differential equation of P_k. Furthermore, all equations in (46) can be solved step by step. The results read

$$\begin{aligned}P_k=&\exp\left(\frac{6}{\nu}\int P_0\mathrm{d}\xi\right)\\&\cdot\left[\frac{1}{\nu}\int g_k\exp\left(-\frac{6}{\nu}\int P_0\mathrm{d}\xi\right)\mathrm{d}\xi+B_k\right],\\&(k=1,2,\cdots),\end{aligned}\tag{48}$$

where B_k are arbitrary integral constants.

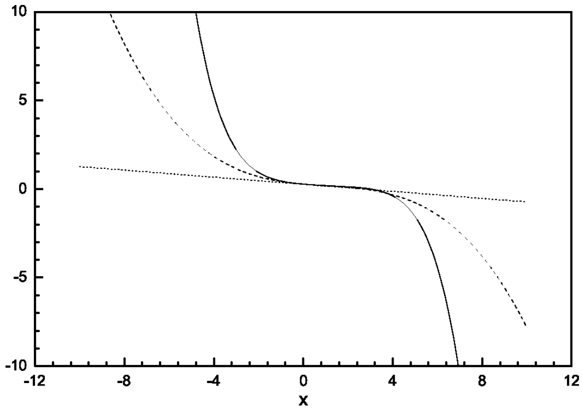


Fig. 4. Plots of truncated series solutions u_0+u_1, $u_0+u_1+u_2$, and $u_0+u_1+u_2+u_3$ of (41) and (49) with $\mu=0.1$, $x_0=\nu=c_{1,0}=c_{1,1}=c_{2,0}=c_{2,1}=c_{3,0}=c_{3,1}=1$, and $a=t=0$.

We can find polynomial solutions for the first four similarity equations in (43):

$$
\begin{aligned}
&P_0=0, \quad P_1=c_{1,0}+c_{1,1}\xi,\\
&P_2=c_{2,0}+c_{2,1}\xi+\frac{3}{\nu}c_{1,0}c_{1,1}\xi^2+\frac{1}{\nu}c_{1,1}^2\xi^3,\\
&P_3=c_{3,0}+c_{3,1}\xi+\frac{3}{\nu^2}(c_{1,1}c_{2,0}\nu+c_{1,1}^2+c_{1,0}c_{2,1}\nu)\xi^2\\
&\qquad+\frac{2}{\nu^2}c_{1,1}(c_{2,1}\nu+3c_{1,0}^2)\xi^3+\frac{6}{\nu^2}c_{1,0}c_{1,1}^2\xi^4\\
&\qquad+\frac{6}{5\nu^2}c_{1,1}^3\xi^5,
\end{aligned} \tag{49}
$$

with $a=A_0=0, A_1=-c_{1,1}\nu, A_2=3c_{1,0}^2-c_{2,1}\nu, A_3=6c_{1,0}c_{2,0}-c_{3,1}\nu+\frac{6}{\nu}c_{1,0}c_{1,1}$, and $c_{1,0}$, $c_{1,1}$, $c_{2,0}$, $c_{2,1}$, $c_{3,0}$, and $c_{3,1}$ are arbitrary constants.

Figure 4 demonstrates the truncated series solutions u_0+u_1 (dotted line), $u_0+u_1+u_2$ (dashed line), and $u_0+u_1+u_2+u_3$ (solid line) when we take $\mu=0.1$, $x_0=\nu=c_{1,0}=c_{1,1}=c_{2,0}=c_{2,1}=c_{3,0}=c_{3,1}=1$, $t=0$ from (41) and (49). It is easily seen that the series solution (42) with respect to all powers of ξ is divergent except that $|\xi|$ is small enough. The singularity of infinite point for the series solution (42) is an essential singularity when t is specified.

4. Conclusion and Discussion

In summary, by applying the approximate symmetry reduction approach to (1+1)-dimensional KdV-Burgers equation under the condition of weak dispersion and weak dissipation, we found that the similarity reduction solutions and similarity reduction equations of different orders are coincident in their forms. Therefore, the series reduction solutions and general formulas for the similarity equations are summarized.

For the weak dissipation case, the zero-order similarity solutions are equivalent to Painlevé II, Painlevé I type, and Jacobi elliptic function solutions. For the weak dispersion case, the zero-order similarity solutions are in the form of Kummer function, Airy function, and hyperbolic tangent function solutions.

k-order similarity reduction equations are linear variable coefficients ordinary differential equations with respect to $P_k(\xi)$. Especially, for the period solutions (expressed by Jacobi elliptic functions) with solitary waves as a special case under weak dissipation, Airy function and hyperbolic tangent function solutions under weak dispersion, higher-order similarity solutions are expressed as general formulas.

We investigated truncated series solutions graphically from the polynomial solutions of similarity reduction equations. For symmetry reduction of the Painlevé I solutions under weak dissipation, approximation of the exact solutions can be improved by increasing terms in the truncated series solutions. For symmetry reduction of the Kummer function solutions under weak dispersion, larger value of t means wider convergence region for the truncated series solutions.

The approximate symmetry reduction approach can be used to search for similar results of other perturbed nonlinear differential equations and it is worthwhile to summarize a general principle for the perturbed nonlinear differential equations holding analogous results.

Acknowledgement

The work was supported by the National Natural Science Foundations of China (Nos. 10735030, 10475055, 10675065 and 90503006), National Basic Research Program of China (973 Program 2007CB814800) and PCSIRT (IRT0734), the Research Fund of Postdoctoral of China (No. 20070410727) and Specialized Research Fund for the Doctoral Program of Higher Education (No. 20070248120).

[1] P.J. Olver, Applications of Lie Group to Differential Equations, second ed., Graduate Texts of Mathematics, Vol. 107, Springer, New York 1993.
[2] G.W. Bluman and J.D. Cole, Similarity Methods for Differential Equations, Appl. Math. Sci. Vol. 13, Springer, Berlin 1974.
[3] G.W. Bluman and S. Kumei, Symmetries and Differential Equations, Appl. Math. Sci. Vol. 81, Springer-Verlag, Berlin 1989.
[4] J.D. Cole, Perturbation Methods in Applied Mathematics, Blaisdell Publishing Company, Waltham Massachusetts 1968.
[5] M. Van Dyke, Perturbation methods in fluid mechanics, CA: Parabolic Press, Stanford 1975.
[6] A.H. Nayfeh, Perturbation Methods, John Wiley and Sons, New York 2000.
[7] V.A. Baikov, R.K. Gazizov, and N.H. Ibragimov, Mat. Sb. **136**, 435 (1988) (English Transl. in: Math. USSR Sb. **64**, 427 (1989)).
[8] V.A. Baikov, R.K. Gazizov, and N.H. Ibragimov, Approximate transformation groups and deformations of symmetry Lie algebras, In: N.H. Ibragimov (ed.), CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3, CRC Press, Boca Raton, FL 1996.
[9] W.I. Fushchich and W.M. Shtelen, J. Phys. A: Math. Gen. **22**, L887 (1989).
[10] M. Pakdemirli, M. Yurusoy, and I.T. Dolapci, Acta Appl. Math. **80**, 243 (2004).
[11] R. Wiltshire, J. Comput. Appl. Math. **197**, 287 (2006).
[12] R.S. Johnson, J. Fluid Mech. **42**, 49 (1970).
[13] L. van Wijngaarden, Ann. Rev. Fluid Mech. **4**, 369 (1972).
[14] G. Gao, Sci. Sinica (Ser A) **28**, 616 (1985).
[15] S.D. Liu and S.K. Liu, Sci. Sinica (Ser A) **35**, 576 (1992).
[16] W. Malfliet and W. Hereman, Phys. Scr. **54**, 569 (1996).
[17] T.R. Marchant and N.F. Smyth, Math. Comput. Model. **28**, 45 (1998).
[18] M.A. Allen and G. Rowlands, J. Plasma Phys. **64**, 475 (2000).
[19] S.Y. Lou, J. Math. Phys. **41**, 6509 (2000).